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Unsteady lifting-line theory by the method of matched asymptotic expansions

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An unsteady lifting-line theory is presented for a general motion of a wing of high aspect ratio. Our matched-asymptotic-expansions analysis parallels that of Van Dyke (1964) in his solution for the steady lifting line, but is complicated by the shedding of transverse vortices associated with variation of circulation with time. The principal result is an expression for the downwash due to three-dimensional effects. Numerical calculations are presented for a wing of elliptic planform following a curved path.

1. Introduction

In terms of computer time it can be very expensive to run a numerical scheme to solve the boundary-value problem for the potential due to inviscid attached flow past a sharp-edged wing, and purely theoretical studies are always desirable, whether as a check on numerical results in simple cases, or for finding general qualitative behaviour.

When the wing is of high aspect ratio a number of approaches are possible. For example, a simple strip theory, while possibly accounting for unsteadiness, cannot correctly predict three-dimensional effects. Conversely, a quasi-steady theory may introduce finiteness of span but will suffer from errors due to vortex shedding.

Recently there have been several attempts to apply the method of matched asymptotic expansions to extend Prandtl's lifting-line theory (Van Dyke 1964) to unsteady flows. With the exception of that of Ahmadi & Widnall (1985) these unsteady lifting-line theories have been inadequate, being either incorrect or based upon invalid assumptions. Betteridge & Archer (1974) consider flapping flight with harmonically oscillating local circulation, but assume a quasi-steady downwash – that for a non-flapping wing with the same circulation distribution. Ahmadi & Widnall (1985) have pointed out that the theory of James (1975), for a motion of the wing with arbitrary forward speed, is incorrect in that his expression for the unsteady induced downwash is infinite since it contains a singularity which should have been removed. Van Holten (1976) has considered a harmonically time-varying motion (relevant to wing flutter or the rotation of a helicopter rotor blade) but assumes that the induced downwash is constant across the chord; this has been shown by Ahmadi & Widnall (1985) to be incorrect. The numerical solution of Phlips, East & Pratt (1981) for the forward flight of flapping birds also only applies to harmonic time-varying motion, and replaces the grid of streamwise and transverse vortices with a simpler structure that is suggested by the interaction of the vortices: a far wake of discrete vortices and a partly rolled-up near wake. Another biological application is examined by Cheng & Murillo (1984) who consider the problem of lunate-tail swimming. This is modelled by a lifting line in harmonic oscillation, including the effects of swept planform. The lifting line theory of Dragoş (1985) (for uniform motion with harmonic time-varying circulation in compressible flow) gives a singular integral equation for the local circulation. This equation could be reduced to quadratures in the same manner that Prandtl's steady theory reduces to quadratures (Van Dyke 1964). Dragoş does not complete his analysis by including calculation of downwash or lift.

Ahmadi & Widnall (1985) have corrected the work of James and Van Holten, and have found the induced downwash and lift and moment coefficients. Unfortunately, they only consider harmonic time-variation of the local circulation. When the motion is in a straight line this is no limitation, since the angle of attack may be split up into its Fourier components (for periodic motion) or represented by its Fourier transform in time (non-periodic motion). However, this is not possible when the path is curved or if the velocity is also varying. The theory of Ahmadi & Widnall is formulated in terms of an acceleration potential (essentially the pressure, p) which, since the problem has been linearized, satisfies Laplace's equation. One advantage of this method is that the potential is continuous everywhere except across the wing (and the lift is given by integrating the pressure jump). However, there is the disadvantage that the solution is not unique since there are eigensolutions with $\partial p/\partial n = 0$ on the wing. Uniqueness is then achieved by integrating the potential from infinity to some point on the wing.

In this paper we shall derive an unsteady lifting-line theory to allow for the more general non-planar motion of a wing of high aspect ratio. Our matched-asymptotic-expansions analysis will parallel that of Van Dyke (1964) in his solution for the steady lifting line, but is complicated by the shedding of transverse vortices associated with variation of circulation with time. By considering the problem in terms of the velocity potential the only question of uniqueness can be decided by applying the unsteady Kutta condition at the trailing edge (Crighton 1985). Although the theory to be presented here is very general as far as the wing motion is concerned, ultimately it must be extended to include the effects of rolling, yawing and of swept planform, as well as the effects of compressibility (the case of an oblique lifting line in steady transonic flow has been examined by Cheng & Meng 1980).

In §2 the problem is stated mathematically, non-dimensionalized and scaled. Since the wing to be considered will be of high aspect ratio, we may make the reasonable lifting-line-theory assumption that spanwise variations in flow are small compared with the wing velocity, and this assumption is crucial to our analysis. Limits on the accuracy of the solution are imposed by the assumption that the wake is not permitted to curl up under the action of local velocities.

In §3 we find the inner limit for the potential where the problem becomes two-dimensional. The outer limit is found in §4, and matching of the two solutions is carried out in §5. This matching introduces three-dimensional effects into the inner problem which cannot be predicted by strip theory. In §5 we present our main result, which gives the effect of spanwise variation in circulation on the downwash when the wing moves in a general manner. Comparison with the work of Ahmadi & Widnall (1985) is made when the wing has constant forward speed but harmonically varying angle of attack. Two further examples are given: the impulsive start of a wing from rest, and the motion of a wing along a curved path. In §6 we consider in more detail the effects of wing thickness on the induced downwash. In §7 Kutta condition is satisfied at the trailing edge and an expression for the lift on the wing is found.

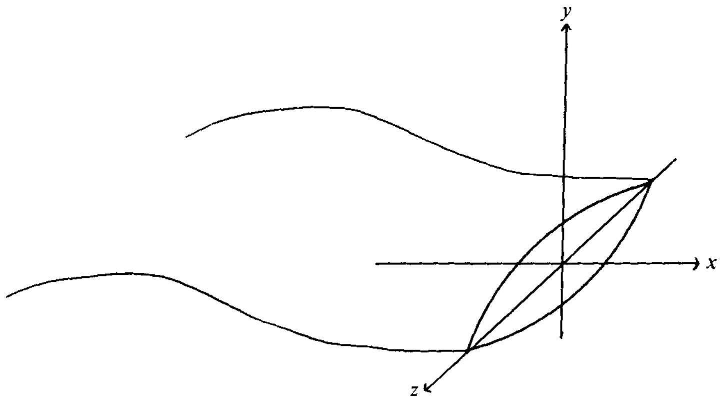

FIGURE 1. Definition sketch.

2. Statement of the problem

We shall first consider a wing of arbitrary cross-section and planform, with span $2l$ and maximum chord c, moving through a fluid at rest at infinity. The flow will everywhere be attached to the wing. The aspect ratio will be large, so that $A = l/c \gg 1$. Axes x, y, z moving with the wing are defined in figure 1, and the wing is permitted to move with three degrees of freedom: horizontal distance x; vertical distance y; angle of attack.

Non-dimensional coordinates are defined by $\bar{x} = x/l$, $\bar{y} = y/l$, $\bar{z} = z/l$ and $\bar{t} = \Omega t$, where Ω is a typical frequency of the motion, and we shall assume that velocities are $O(l\Omega)$. Since we have scaled with span, the limiting case as $A \to \infty$ is a finite-span wing with $O(A^{-2})$ chord and with velocity potential and circulation being $O(A^{-1})$ and $O(A^{-2})$ respectively.

The coordinates of the centreline of the wing with respect to axes fixed in space (i.e. fixed with respect to the fluid at infinity) will be taken to be $(\xi(\bar{t}), \zeta(\bar{t}))$.

Cheng (1975) has identified five regimes for the frequency of the motion:

$$\text{I} \qquad c \ll l \ll \frac{2\pi U}{\Omega},$$

$$\text{II} \qquad c \ll l = O\left(\frac{2\pi U}{\Omega}\right),$$

$$\text{III} \qquad c \ll \frac{2\pi U}{\Omega} \ll l,$$

$$\text{IV} \qquad c = O\left(\frac{2\pi U}{\Omega}\right) \ll l,$$

$$\text{V} \qquad \frac{2\pi U}{\Omega} \ll c \ll l,$$

where U is a typical wing velocity. In regime I, of very low frequencies, a quasi-steady theory is adequate, unsteady effects being of a smaller order than three-dimensional effects. In regimes II and IV the problem requires analysis over two lengthscales, c and l, whereas in regime III three distinct regions of space exist. In regime V an averaging of the problem over time is possible. Our analysis will be concerned with domain II.

For an inviscid, incompressible, irrotational flow we have the existence of a velocity potential ϕ except on the trailing wake where $\nabla\phi$ will be discontinuous. Non-dimensionalizing, we choose $\bar{\phi} = \phi/\Omega l^2$. The problem to be solved is

$$\bar{\phi}_{\bar{x}\bar{x}} + \bar{\phi}_{\bar{y}\bar{y}} + \bar{\phi}_{\bar{z}\bar{z}} = 0$$

everywhere in the fluid except on the wake, with $\nabla\phi \rightarrow 0$ at infinity and also the Kutta condition, that $\nabla\bar{\phi}$ is to be finite, at the trailing edge. On the surface of the wing there is to be no normal flow.

With the assumption of small spanwise variation in flow compared with the wing velocity we may take advantage of the large aspect ratio and define inner coordinates $x' = A\bar{x}$, $y' = A\bar{y}$, and an inner potential $\phi'(x', y', \bar{z})$. In these coordinates the inner potential must satisfy

$$\phi'_{x'x'} + \phi'_{y'y'} + A^{-2}\phi'_{\bar{z}\bar{z}} = 0$$

and so the first two terms of an expansion of ϕ' in powers of A^{-1} will be two-dimensional, with $\bar{z}$ as a parameter.

We shall define the cross-section of the wing in a plane $\bar{z}$ = constant, by the conformal transformations $x' + iy' = s' = e^{i\gamma(\bar{t})}s$, $s = f(w)$ and $|w| = 1$, where the points at infinity in the s- and w-planes coincide, that is

$$s \sim a_{-1}w + a_0 + \frac{a_1}{w} + \frac{a_2}{w^2} + \dots,$$

where a_{-1} is real and positive. The coefficients a_n will be functions of the spanwise coordinate $\bar{z}$. We have thus mapped the unit disk onto the wing cross-section. $\gamma(\bar{t})$ is the angle of pitch that the wing makes with a frame of reference fixed in space, and is therefore related to the angle of attack (which also depends upon the direction of motion of the wing). The inner problem will be non-uniform (to some order) at the wing tips, and so there will be a constraint on the decay of the coefficients a_n towards the wing tip; this will be discussed later.

In order for a circulation to be set up around the wing there must be a sharp trailing edge, so that $df/dw = 0$ at some point of $|w| = 1$; we shall choose this point to be at $w = -1$ without loss of generality.

We have mentioned that the inner limit as $A \rightarrow \infty$ is that of two-dimensional potential flow. The outer problem, at distances from the wing of the order of the span, then becomes that of a line of singularities (a loaded line) and this is shown schematically in figure 2. Our method will follow that of Van Dyke (1964) with modifications to allow for shedding of transverse vortices due to the unsteady motion of the wing (these vortices are shown in figure 2 as lines parallel to the centreline of the wing).

Before proceeding to derive the inner and outer solutions we shall comment on the trailing wake. This wake will be convected with the total flow and therefore it will be necessary, in order to make the problem tractable, to assume a position for the wake that may not be its true one, and in so doing limit the accuracy to which our solution is valid. This is a common assumption; for example, when the wing moves along the x-axis the wake is often assumed to lie on $y = 0$ (e.g. Van Dyke 1964). Since, as will be seen shortly, we shall need to find two terms in the inner expansion that contain expressions due to the wake, then we must, for consistency, know the position of the wake to second order in A^{-1}. However, since lengths have been scaled with span and not chord, the circulation about the wing is $O(A^{-1}\theta)$ where θ is the angle of attack, and over an $O(1)$ timescale (the timescale of interest to us) distortion of the wake due to interaction with itself will be over distances of $O(A^{-1}\theta)$. We shall therefore only consider motion with angle of attack $O(A^{-1})$. This is not a very

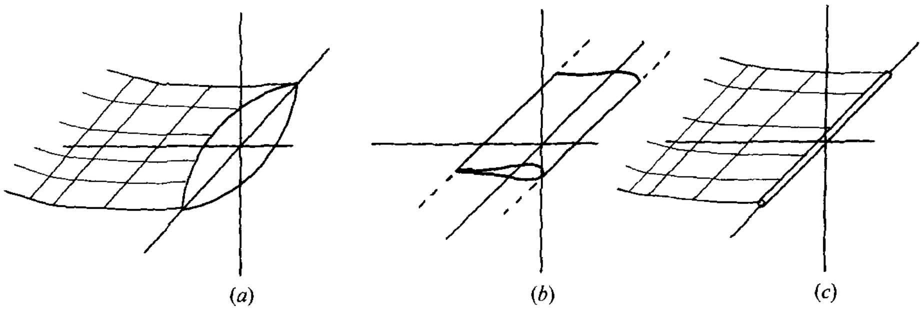

FIGURE 2. (a) Full problem. (b) Inner two-dimensional limit. (c) Outer limit of a loaded line.

restrictive assumption, since the following is in any case not valid when the angle of attack is within the stall region. With $O(A^{-1})$ angle of attack the velocities created by the wake are $O(A^{-2})$ compared with the $O(1)$ velocities due to uniform flow past the body.

We assume the outer limit to have an asymptotic expansion of the form

$$\bar{\phi} \sim A^{-2}\bar{\phi}_2 + o(A^{-2}),$$

since the local circulation is $O(A^{-2})$, and the inner limit to have the form

$$\phi' \sim A^{-1}\phi_1' + A^{-2}\phi_2' + A^{-3}\phi_3' + o(A^{-3}),$$

where terms involving $\ln A$ have not yet been included since their positions will not be known until matching occurs.

3. The inner problem – vortex shedding behind an aerofoil

Henceforth, overbars will be dropped from scaled variables; the primes will be retained to distinguish inner variables. An overbar will now denote complex conjugation.

In the inner variables we have

$$\phi'_{x'x'} + \phi'_{y'y'} + A^{-2}\phi'_{zz} = 0, \tag{3.1}$$

with no flow normal to the wing surface. We shall generalize the method of Isaacs (1945) (for the unsteady motion of an aerofoil along a straight path) in the following. First we consider the two-dimensional potential due to a wake of vortices shed from the trailing edge of the wing.

If the strength of the local circulation around the wing is $\Gamma(t,z)$ then at time τ the wing sheds a vortex of strength $-\Gamma(\tau,z)\,\delta\tau$. At time t this vortex will be at $s'_*(t,\tau)$ in the s' plane and at $w_*(t,\tau)$ in the w-plane (figure 3). By Milne-Thomson's (1938) circle theorem the complex potential due to the wake of vortices shed from time t_0 (the time at which motion begins) is the superposition of vortices in the wake and their images in the circle, and may be written as

$$\mathrm{i}\int_{t_0^-}^{t} \Gamma_\tau(\tau,z)\ln\left\{\frac{w-w_*(t,\tau)}{1-w\bar{w}_*(t,\tau)}\right\}\mathrm{d}\tau + \mathrm{i}\Gamma(t,z)\ln w$$

(assuming $\Gamma(t_0^-,z)=0$) with subscript τ denoting a partial derivative with respect to τ. This potential satisfies the condition of no normal flow.

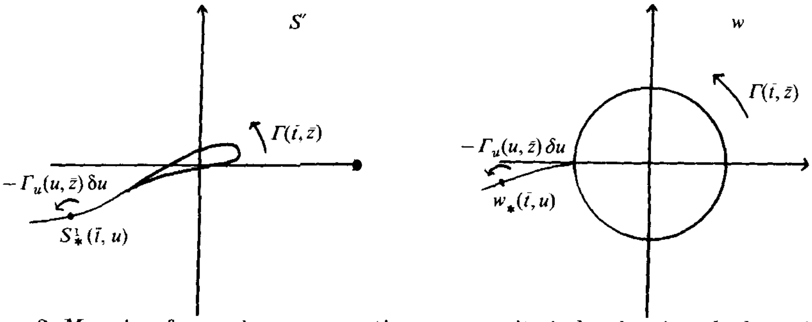

FIGURE 3. Mapping from wing cross-section on to unit circle, showing shed vorticity.

Including this term, the complete inner velocity potential for attached flow is given by the real part of

$$\begin{aligned}W \sim\ & A^{-1}\left\{\bar{I}_1 s - a_{-1}\left(\bar{I}_1 w + \frac{I_1}{w}\right)\right\} + A^{-2}\mathrm{i}\dot{\gamma}N(w) \\ & + A^{-2}\frac{\mathrm{i}}{2\pi}\int_{t_0^-}^{t} \Gamma_{2_\tau}(\tau, z)\ln\left\{\frac{w-w_*}{1-w\bar{w}_*}\right\}\mathrm{d}\tau \\ & + A^{-3}\frac{\mathrm{i}}{2\pi}\int_{t_0^-}^{t} \Gamma_{3_\tau}(\tau, z)\ln\left\{\frac{w-w_*}{1-w\bar{w}_*}\right\}\mathrm{d}\tau \\ & + A^{-3}\left\{\bar{I}_3 w + \frac{I_3}{w}\right\} + A^{-3}W^*(w,\bar{w}) + o(A^{-3}),\end{aligned}$$

where $\dot{} = \mathrm{d}/\mathrm{d}t$ and functions of z and t alone have been dropped. Here $I_1 = \mathrm{e}^{-\mathrm{i}\gamma}(\dot{\xi}+\mathrm{i}\dot{\zeta})$ is a function of t, and I_3, which is a function of z and t, comes from matching with the outer potential ϕ_2. Note that the angle of attack is $-\arg(I_1)$. The $O(A^{-2})$ and $O(A^{-3})$ local circulations (Γ_2 and Γ_3 respectively) may be defined uniquely by invoking the unsteady Kutta condition of regular flow in the vicinity of the trailing edge (Crighton 1985). The term $A^{-2}\mathrm{i}\dot{\gamma}N(w)$ represents the potential due to a rotation of the wing. $N(w)$ consists of the negative powers of w in the Laurent expansion of $f(w)\bar{f}(w^{-1})$ (Milne-Thomson 1938). The term W^* is a solution of the two-dimensional Poisson equation (with z as a parameter) which comes from expanding (3.1) in an asymptotic series in powers of A. If we denote the inverse of the transformation f by F then $W^*(F(s), \overline{F(s)})$ is to satisfy

$$\mathrm{Re}\{4W^*_{s\bar{s}} + W_{1_{zz}}\} = 0$$

and the condition of no normal flow on $|w| = 1$, where

$$W_1 = \bar{I}_1 s - a_{-1}\left(\bar{I}_1 w + \frac{I_1}{w}\right) - \bar{I}_1 a_0,$$

remembering that a_n is a function of z. The second z-derivative term is due to the third term in (3.1). W_1 is simply the $O(A^{-1})$-term in the inner potential with $-a_0\bar{I}_1$ being one of the functions of z and t that were dropped above.

Effects of the three-dimensionality of the problem will appear in the terms I_3 and W^*, as well as Γ_3 via the Kutta condition. These terms will therefore not be correctly predicted by strip theory.

To avoid complications due to the distortion of the wake under the action of local velocities we shall approximate the position of the wake by the path of the trailing edge (and this approximation improves as the wing becomes thin and as the angle of attack tends to zero); thus

$$f(w_*(t,\tau)) = \mathrm{e}^{-\mathrm{i}\gamma(t)}\{\mathrm{e}^{\mathrm{i}\gamma(\tau)}f(-1)+A(\chi(t)-\chi(\tau))\},$$

where $\chi(t) = \xi(t)+\mathrm{i}\zeta(t)$. The A appears because in inner variables the centre of the wing is at $(A\xi(t), A\zeta(t))$. The inner complex potential then becomes

$$A^{-1}\left\{\bar{I}_1 s - a_{-1}\left(\bar{I}_1 w+\frac{I_1}{w}\right)\right\}+A^{-2}\mathrm{i}\dot{\gamma}N(w)+A^{-2}\frac{\mathrm{i}}{2\pi}\int_{t_0^-}^{t}\{\Gamma_{2_\tau}+A^{-1}\Gamma_{3_\tau}\}$$
$$\times\ln\left\{\frac{w-F[\mathrm{e}^{-\mathrm{i}\gamma(t)}(\mathrm{e}^{\mathrm{i}\gamma(\tau)}f(-1)+A(\chi(t)-\chi(\tau)))]}{1-w\bar{F}[\mathrm{e}^{\mathrm{i}\gamma(t)}(\mathrm{e}^{-\mathrm{i}\gamma(\tau)}\overline{f(-1)}+A(\bar{\chi}(t)-\bar{\chi}(\tau)))]}\right\}\mathrm{d}\tau$$
$$+A^{-3}\left\{\bar{I}_3 w+\frac{I_3}{w}\right\}+A^{-3}W^*+o(A^{-3}).$$

As yet this expression is not in terms of powers and logarithms of A and we cannot yet match the inner and outer potentials. However, by splitting the integral into two parts, from t_0^- to $t-\nu$ and from $t-\nu$ to t, where $A^{-1} \ll \nu \ll 1$, the integral may be expanded for large A to give

$$W \sim A^{-1}\left\{\bar{I}_1 s-a_{-1}\left(\bar{I}_1 w+\frac{I_1}{w}\right)\right\}+A^{-2}\mathrm{i}\dot{\gamma}N(w)-A^{-2}\frac{\mathrm{i}}{2\pi}\Gamma_2\ln w-(A^{-3}\ln A)$$
$$\times\frac{\mathrm{i}}{2\pi}a_{-1}\Gamma_{2_\tau}\left\{\frac{w\,\mathrm{e}^{\mathrm{i}\gamma(t)}}{\dot{\chi}(t)}-\frac{\mathrm{e}^{-\mathrm{i}\gamma(t)}/w}{\dot{\bar{\chi}}(t)}\right\}-(A^{-3}\ln A)\frac{\mathrm{i}}{2\pi}\Gamma_{3_a}\ln w$$
$$-A^{-3}a_{-1}\frac{\mathrm{i}}{2\pi}\int_{t_0^-}^{t}\left[\Gamma_{2_\tau}\left\{\frac{w\mathrm{e}^{\mathrm{i}\gamma(t)}}{\chi(t)-\chi(\tau)}-\frac{\mathrm{e}^{-\mathrm{i}\gamma(t)}/w}{\bar{\chi}(t)-\bar{\chi}(\tau)}\right\}\right.$$
$$\left.-\Gamma_{2_t}\left\{\frac{w\,\mathrm{e}^{\mathrm{i}\gamma(t)}}{\dot{\chi}(t)}-\frac{\mathrm{e}^{-\mathrm{i}\gamma(t)}/w}{\dot{\bar{\chi}}(t)}\right\}\frac{1}{t-\tau}\right]\mathrm{d}\tau$$
$$+A^{-3}\frac{\mathrm{i}}{2\pi}\Gamma_{2_t}\int_0^\infty\left[\ln\left\{\frac{w-h(\tau,t)}{1/w-\bar{h}(\tau,t)}\right\}-\ln\left\{\mathrm{e}^{-2\mathrm{i}\gamma(t)}\frac{\dot{\chi}(t)}{\dot{\bar{\chi}}(t)}\right\}+H(-t+t_0)\frac{a_{-1}}{\tau}\right.$$
$$\left.\times\left\{\frac{w\mathrm{e}^{\mathrm{i}\gamma(t)}-(\mathrm{e}^{\mathrm{i}\gamma(t)}f(-1)-a_0)/a_{-1}}{\dot{\chi}(t)}-\frac{\mathrm{e}^{-\mathrm{i}\gamma(t)}/w-(\mathrm{e}^{-\mathrm{i}\gamma(t)}\overline{f(-1)}-\bar{a}_0)/a_{-1}}{\dot{\bar{\chi}}(t)}\right\}\right]\mathrm{d}\tau$$
$$-A^{-3}\frac{\mathrm{i}}{2\pi}\Gamma_3\ln w+A^{-3}\left\{\bar{I}_3 w+\frac{I_3}{w}\right\}+A^{-3}W^*+o(A^{-3}) \tag{3.2}$$

(details of this expansion may be found in Wilmott 1985) where H is the unit Heaviside function and

$$h(\tau,t) = F[f(-1)+\tau\,\mathrm{e}^{-\mathrm{i}\gamma(t)}\dot{\chi}(t)].$$

The argument of F represents an approximation to the position of the wake that is sufficiently accurate for our purposes.

Because of the appearance of a term of $O(A^{-3}\ln A)$ we have introduced the 'switchback' bound-vortex term (Lagerstrom & Casten 1972) $(A^{-3}\ln A)(\mathrm{i}/2\pi)\Gamma_{3_a}\ln w$ in order to satisfy the Kutta condition.

We shall now comment briefly on the problem for W^*. For a wing at $O(A^{-1})$ angle of attack $\bar{I}_1-I_1 = O(A^{-1})$, and since $\bar{I}_1 s$ is independent of z then to first order W^* satisfies

$$W^*_{s\bar{s}} = -\frac{1}{4}\frac{\partial^2}{\partial z^2}\bigg|_{s\text{ fixed}}\left\{a_{-1}I_1\left(w+\frac{1}{w}\right)-a_0 I_1\right\}. \tag{3.3}$$

To a particular solution of this must be added a function of w and a function of $\bar{w}$ to satisfy the tangency condition. When calculating Γ_3 it is necessary to know W^* in order to satisfy the Kutta condition. This problem has been encountered by Van Dyke (1964) in his work on steady lifting-line theory, for a wing in an infinite fluid, since it is not just a feature of unsteady flow. The solution for W^* for a general wing requires knowledge of the mapping $f(w)$. However, for a wing that is nearly flat, so that $s \sim a_{-1}(z)(w+1/w)+O(A^{-1})$, then $W^* = O(A^{-1})$. Henceforth, except in §6, in order not to obscure more important details we shall only consider thin wings so that we may take $W^* = 0$. For such thin wings we also have $N(w) = A^{-1}N^*(w)$ where N^* is $O(1)$.

We note that when the wing is indeed thin it is possible to express the coefficient a_{-1} in terms of the wing thickness. This may be done by considering the mapping that takes the slit $[-1,1]$ onto the unit circle, and which therefore maps the nearly flat wing cross-section onto a nearly circular closed curve (Nehari 1952). This is equivalent to Goldstein's (1952) thin-aerofoil approximation to Theodorsen's (1932) theory.

We have now found the inner potential to $O(A^{-3})$, subject to matching with the outer potential. Note that the $O(A^{-4})$-terms will be affected by wake curl-up, and our assumption about the position of the trailing wake is not sufficiently accurate to justify finding these terms.

4. The outer problem – the lifting-line potential

Writing (3.2) in terms of the outer coordinates we find that ϕ_2 is to match with two-dimensional vortices of strength $\Gamma_2(t,z)$, $(\Gamma_2/2\pi)\tan^{-1}y/x$ being the leading term in the expansion of the real part of $-i\Gamma_2/2\pi\ln(w)$, and two-dimensional dipoles of strength $a_{-1}\,\mathrm{Re}\{a_1 e^{2i\gamma}\dot{\bar{\chi}}-a_{-1}\dot{\chi}\}$, being the dipole coefficient to leading order when the W_1 term in W is expanded. Both of these distributions lie along the line $x=y=0$, $-1 \leqslant z \leqslant 1$. The dominant effect is therefore of a line of vortices and dipoles (higher-order multipoles have a higher-order effect on the outer flow). When the wing is thin the dipole effect is small and appears in a higher-order term in the outer potential. In order to perform the matching with the vortices we follow the approach of Dragoş (1985) (for uniform motion with harmonic time-variation of the circulation) but generalize the method to arbitrary velocities and circulations.

In moving outer coordinates the Euler equation and continuity equations become

$$\boldsymbol{q}_t-\dot{\xi}\boldsymbol{q}_x-\dot{\zeta}\boldsymbol{q}_y+(\boldsymbol{q}\cdot\boldsymbol{\nabla})\,\boldsymbol{q} = -\boldsymbol{\nabla}p, \tag{4.1}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{q} = 0, \tag{4.2}$$

where $\Omega l\boldsymbol{q}$ is the fluid velocity and $\rho\Omega^2l^2p$ the pressure. These equations are to hold everywhere except (i) on the loaded line that is the outer limit of the wing of high aspect ratio as $A\rightarrow\infty$, and (ii) on the trailing wake across which the conditions $[p]=[\boldsymbol{q}\cdot\boldsymbol{n}]=0$ are to hold, $\boldsymbol{n}$ being the normal to the wake. In the outer region, the effect of the circulation around the wing on the velocities is $O(A^{-2})$ and so we may linearize (4.1) to give

$$\boldsymbol{q}_t-\dot{\xi}\boldsymbol{q}_x-\dot{\zeta}\boldsymbol{q}_y = -\boldsymbol{\nabla}p.$$

Matching with a distribution of vorticity along the z-axis may be ensured by modifying this equation of motion to

$$\boldsymbol{q}_t-\dot{\xi}\boldsymbol{q}_x-\dot{\zeta}\boldsymbol{q}_y = -\boldsymbol{\nabla}p+\Gamma_2(t,z)\,\delta(x)\,\delta(y)\,(\dot{\zeta},-\dot{\xi},0), \tag{4.3}$$

$(\dot{\zeta}, -\dot{\xi}, 0)$ being the vector perpendicular to the direction of motion. That is, a lifting line may be modelled by a line force (represented by the delta functions) in the momentum equation. This is not surprising since the circulation produces a lift on the wing and hence a force on the fluid in the opposite direction. This result can be shown to be true by multiplying (4.2) by a test function, $\psi(x, y; z)$, and integrating over a plane $z =$ constant. After substituting the local expression for $\boldsymbol{q}$, the velocity due to a two-dimensional vortex,

$$\boldsymbol{q} = \frac{\Gamma_2(t,z)}{2\pi(r+\epsilon_n)^2} \quad (-y, x, 0)$$

(which has been smoothed to be finite everywhere) we may apply Green's theorem, and in the limit $\epsilon_n \to 0$ get the result

$$\iint_{\mathbb{R}^2} \psi(\boldsymbol{q}_t - \dot{\xi}\boldsymbol{q}_x - \dot{\zeta}\boldsymbol{q}_y + \nabla p)\,\mathrm{d}x\,\mathrm{d}y = \psi(0,0;z)\,\Gamma_2(t,z)\,(\dot{\zeta}, -\dot{\xi}, 0),$$

where $\mathbb{R}^2$ is the two-dimensional cross-section parallel to the (x, y)-plane. A weak solution for $\boldsymbol{q}$ satisfies (4.1) and (4.2) where differentiable, and the jump conditions $[p] = [\boldsymbol{q}\cdot\boldsymbol{n}] = 0$ across planes of discontinuity. We therefore have that a weak solution for $\boldsymbol{q}$ satisfies (4.3) and the continuity equation. This result is discussed further in Ockendon & Wilmott (1986). Before solving (4.2) and (4.3) in the general case it will be helpful to consider the simpler example having $\dot{\xi} = 1$, $\dot{\zeta} = 0$ and $\Gamma_2 = \Gamma(z)$. Taking the Fourier transform of (4.2) and (4.3) in this case yields $\boldsymbol{q} = \nabla\phi$ where

$$\phi = \frac{1}{4\pi}\int \frac{\Gamma(\eta)\,y}{y^2+(z-\eta)^2}\left\{1 - \frac{x}{[x^2+y^2+(z-\eta)^2]^{\frac{1}{2}}}\right\}\mathrm{d}\eta,$$

which is the Prandtl lifting-line potential (Van Dyke 1964).

In the most general case, when ξ and ζ are arbitrary functions of t, and Γ_2 depends upon both t and z, then (4.2) and (4.3) may be readily solved by Fourier-transform methods to give $\boldsymbol{q} = \nabla\phi$, where

$$\phi = \frac{-1}{4\pi}\int_{-1}^{1}\int_{t_0^-}^{t} \frac{\Gamma_2(\tau,\eta)[\dot{\zeta}(\tau)\,(x+\xi(t)-\xi(\tau)) - \dot{\xi}(\tau)\,(y+\zeta(t)-\zeta(\tau))]}{[(x+\xi(t)-\xi(\tau))^2+(y+\zeta(t)-\zeta(\tau))^2+(\eta-z)^2]^{\frac{3}{2}}}\,\mathrm{d}\tau\,\mathrm{d}\eta. \qquad (4.4)$$

This solution has a discontinuous gradient at points $(\xi(\tau)-\xi(t), \zeta(\tau)-\zeta(t), z)$ with $t_0 \leqslant \tau \leqslant t$ and $-1 \leqslant z \leqslant 1$, which generate the surface spanned by the loaded line; the line disturbance has propagated along the material lines of the flow. This potential is seen to be composed of a distribution of infinitesimal vortex rings of strength $\Gamma(\tau, \eta)$, with axes perpendicular to the local direction of motion, positioned at $(\xi(\tau)-\xi(t), \zeta(\tau)-\zeta(t), \eta)$, relative to the wing, for $t_0 \leqslant \tau \leqslant t$ and $-1 \leqslant z \leqslant 1$. If the trailing vorticity (in the flow direction) and the shed vorticity (parallel to the wing span) are imagined to form a grid of vortex lines then the vortex rings are the closed loops formed by two sides of shed and two sides of trailing vorticity; this is an alternative interpretation of the wake structure. We therefore see that as far as the outer limit is concerned, we have automatically catered for the wake and its approximate position.

5. Unsteady induced downwash

In order to find the downwash at the wing due to the trailing wake, we must continue with the matching process. Expanding the inner potential (3.2) in outer variables with $s_1 = \bar{x}+\mathrm{i}\bar{y}$, we have that the outer limit of the inner potential is, to $O(A^{-2})$, the real part of

$$A^{-2}\frac{\mathrm{i}}{2\pi}\Gamma_2 \ln s_1 + \frac{\mathrm{i}A^{-2}\Gamma_{2_t}}{2\pi\dot{\chi}(t)}\{s_1 \ln s_1 - s_1 \ln(\dot{\chi}(t)(t-t_0)) - s_1\}$$
$$-\frac{\mathrm{i}A^{-2}}{2\pi}s_1\int_{t_0^-}^{t}\left[\frac{\Gamma_{2_\tau}(\tau,z)}{\chi(t)-\chi(\tau)} - \frac{\Gamma_{2_t}(t,z)}{(t-\tau)\,\dot{\chi}(\bar{t})}\right]\mathrm{d}\tau + \frac{A^{-2}}{a_{-1}}\bar{I}_3\,\mathrm{e}^{-\mathrm{i}\gamma}\,s_1.$$

The inner limit of the outer potential, $A^{-2}\phi_2$, must next be found in order to perform the matching. Details are not given here, but the procedure can be compared with that of Van Dyke (1964) and a brief outline of the method used in expanding (4.4) can be found in our Appendix. Matching then yields an expression for I_3. The inner potential is now fully known to $O(A^{-3})$, subject only to the satisfaction of the Kutta condition.

The downwash itself is the velocity component of the flow perpendicular to the wing's direction of motion. This component contains a contribution due to finite-span, three-dimensional effects (as well as terms present in a genuinely two-dimensional problem of flow past an aerofoil) and which may be found solely from the inner limit of the outer potential. From this limit (to be found in the Appendix) the velocity component of the flow in the direction $(\dot{\xi}^2+\dot{\zeta}^2)^{-\frac{1}{2}}(\dot{\zeta},-\dot{\xi})$ is seen to be

$$\begin{aligned}
V_{3\mathrm{D}}(t,z) = {} & \frac{1}{4\pi|\dot{\chi}(t)|}\int_{-1}^{1}\left[\frac{\Gamma_{2_\eta}(t,\eta)}{\eta-z}\,\frac{[(\eta-z)^2+|\dot{\chi}(t)|^2(t-t_0)^2]^{\frac{1}{2}}}{t-t_0} + \Gamma_{2_{\eta t}}(t,\eta)\,\operatorname{sgn}(\eta-z)\right.\\
& \left.\times\left\{\ln|\eta-z| + \tfrac{1}{2}\ln\left\{\frac{\eta-z-[(\eta-z)^2+|\dot{\chi}(t)|^2(t-t_0)^2]^{\frac{1}{2}}}{\eta-z+[(\eta-z)^2+|\dot{\chi}(t)|^2(t-t_0)^2]^{\frac{1}{2}}}\right\}\right\}\right]\mathrm{d}\eta\\
& + \frac{1}{4\pi}\int_{-1}^{1}\int_{t_0^-}^{t}\left[\frac{(\eta-z)\,\Gamma_{2_\eta}(\tau,\eta)}{|\chi(t)-\chi(\tau)|^2[|\chi(t)-\chi(\tau)|^2+(\eta-z)^2]^{\frac{1}{2}}}\right.\\
& \times\left\{-\frac{\dot{\chi}(\tau)\,\dot{\bar{\chi}}(t)+\dot{\bar{\chi}}(\tau)\,\dot{\chi}(t)}{2|\dot{\chi}(t)|} - \frac{1}{4|\dot{\chi}(t)|}\{\dot{\bar{\chi}}(\tau)\,(\chi(t)-\chi(\tau)-\dot{\chi}(\tau)\,(\bar{\chi}(t)-\bar{\chi}(\tau))\}\right.\\
& \times\{\dot{\bar{\chi}}(t)\,(\chi(t)-\chi(\tau))-\dot{\chi}(t)\,(\bar{\chi}(t)-\bar{\chi}(\tau))\}\\
& \left.\times\left\{\frac{2}{|\chi(t)-\chi(\tau)|^2}+\frac{1}{[|\chi(t)-\chi(\tau)|^2+(\eta-z)^2]}\right\}\right\}\\
& \left.+\frac{(\eta-z)\{\Gamma_{2_\eta}(t,\eta)+(\tau-t)\,\Gamma_{2_{\eta t}}(t,\eta)\}}{|\dot{\chi}(t)|\,(\tau-t)^2[|\dot{\chi}(t)|^2(\tau-t)^2+(\eta-z)^2]^{\frac{1}{2}}}\right]\mathrm{d}\tau\,\mathrm{d}\eta
\end{aligned}\tag{5.1}$$

(a factor A^{-2} has been excluded). For (5.1) to exist we require $\dot{\xi}$, $\dot{\zeta}$, Γ_{2_z} and $\Gamma_{2_{zt}}$ to be continuous.

It is possible, for sufficiently blunt wing tips, for this downwash to become large and ultimately infinite in a neighbourhood of the tips, therefore invalidating our assumption of small perturbation to the incident flow. The size of this region will depend on the shape of the wing tip, being larger for blunter tips. It is possible to remove such non-uniformities by deriving further asymptotic expansions near the tip which may then be matched with the inner flow. Nevertheless, for parabolic and more slender wing tips the singularity is at most logarithmic, which will result in convergent values for the total wing loading. Of course, a higher-order theory along the present lines, not allowing for the non-uniformity, will lead (at higher orders in A^{-1}) to infinite total lift even for parabolic wing tips. Such wing-tip singularities will only exist in the fully attached flow assumed here. In practice flow separation would occur at the tips and we assume that this would have a small effect on the overall flow and on the downwash and forces in particular. The first two terms in (5.1) depend only upon the instantaneous values of the velocity and circulation, whereas the final term is dependent upon the whole history of the motion. The limit $t_0 \to -\infty$ may be taken in the above, and in the limit of steady flow the only term remaining is

$$\frac{1}{4\pi}\int_{-1}^{1}\frac{\Gamma_{2_\eta}(t,\eta)}{\eta - z}\,\mathrm{d}\eta,$$

which is the well-known result of Prandtl's steady lifting-line theory.

In the case $\dot{\xi} = 1, \zeta = 0, \Gamma_2 = \Gamma^*(z)\,\mathrm{Re}\{\mathrm{e}^{\mathrm{i}\omega t}\}$ (5.1) reduces, after some manipulation, to those terms in (6.16) of Ahmadi & Widnall (1985) (for the downwash due to a wing moving with constant velocity but harmonically varying circulation) that are due to three-dimensional effects.

Applying the Kutta condition that $\mathrm{d}W/\mathrm{d}w$ is to be zero at the trailing edge, $w = -1$, yields, to lowest order

$$\Gamma_2(t,z) = -4\pi a_{-1} A\ \mathrm{Im}\{\mathrm{e}^{-\mathrm{i}\gamma(t)}\dot{\chi}(t)\}, \tag{5.2}$$

where $\gamma(t) - \arg\dot{\chi}(t)$ is the angle of attack.

We are now in a position to consider specific examples.

5.1. *Impulsively accelerated wing*

As our first example we shall consider the impulsive acceleration of the wing from rest along a straight path. The displacement will be given by

$$\xi(t) = \begin{cases} 0 & (t < 0), \\ t & (t > 0), \end{cases}$$
$$\zeta(t) = 0.$$

For a constant angle of attack the lowest-order local circulation Γ_2 is a function of z only. For a wing of elliptic planform with $\Gamma_2 = -(1-z^2)^{\frac{1}{2}}$, the downwash is plotted against spanwise direction in figure 4 for several values of t. This downwash, which is given by the single integral term in (5.1), decreases monotonically in time, approaching, as $t \to \infty$, the constant value given by the steady state. Note that since $\dot{\xi}(t)$ is discontinuous at $t = 0$ the expression for $V_{3\mathrm{D}}$ is only valid for $t \gg A^{-1}$. After the wing has travelled one span length ($t = 2$) the downwash is already only 10% away from its final value.

5.2. *Motion of the wing on a curved path*

Since the main work of this paper is in finding the downwash at the trailing edge due to spanwise variation in circulation without the restriction of constant direction of motion, we shall consider the case of constant angle of attack, with displacement given by

$$(\xi(t), \zeta(t)) = \begin{cases} (t, 1) & (t < 0), \\ (\sin t, \cos t) & (0 < t < \alpha), \\ ((t-\alpha)\cos\alpha + \sin\alpha, (t-\alpha)\sin\alpha + \cos\alpha) & (\alpha < t). \end{cases}$$

This represents a wing moving with constant velocity coming from infinity on a straight path, smoothly turning through an angle α by following the arc of a circle and then returning to infinity. Again the lowest-order circulation Γ_2 is a function of z alone and not of t. This is because the speed and angle of attack are constant and

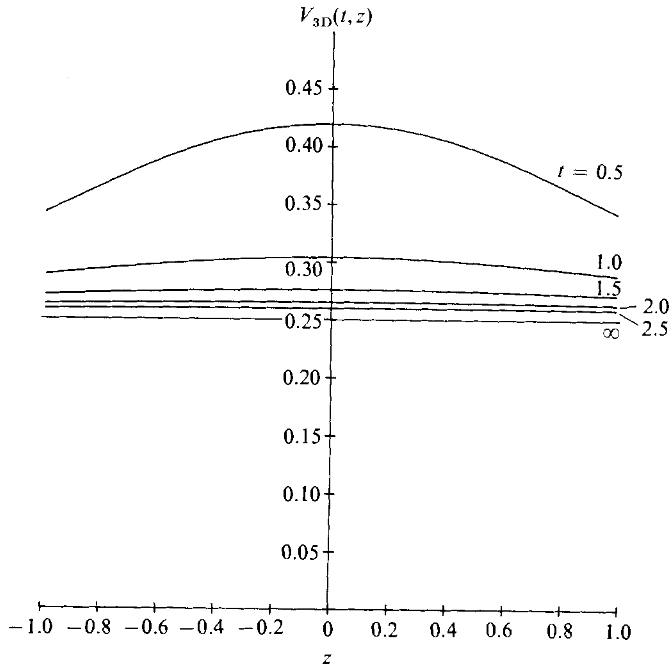

FIGURE 4. Downwash $V_{3D}(t, z)$ against z for impulsively started wing.

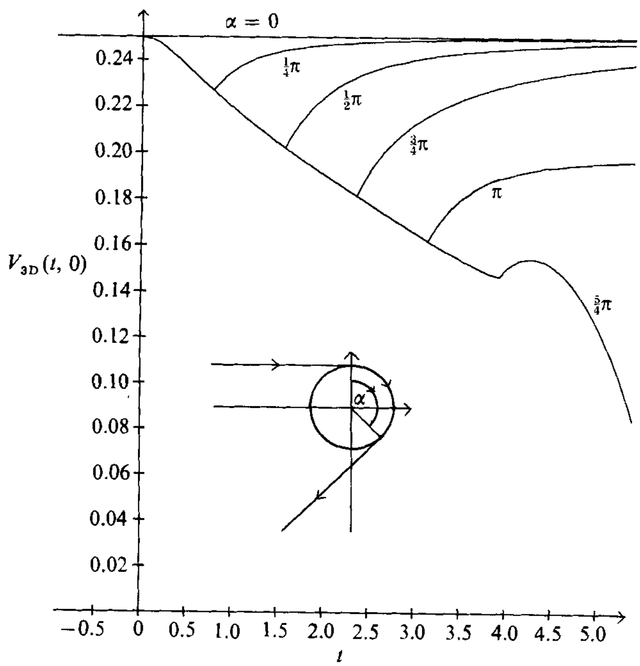

FIGURE 5. Mid-span downwash, $V_{3D}(t, 0)$, against time for wing on the curved path shown in inset.

the radius of the turning circle is large compared with the chord. We shall again consider a wing with an elliptic planform, with $\Gamma_2 = -(1-z^2)^{\frac{1}{2}}$. The mid-span downwash, $V_{3D}(t,0)$ is plotted against time in figure 5 for several values of α. We may note several things about V_{3D}. First, the downwash is decreased due to a turning and this has the effect of increasing the effective angle of attack. This decrease in downwash is due to the vortex wake no longer lying in the plane of flight, that is, the component of downwash due to the longitudinal vortices becomes less as the angle between the wake plane and the flight plane increases. This is also true of transverse vortices, although in the above example these are not present since Γ_2 is independent of time. Secondly, the slope of V_{3D} is discontinuous when the wing leaves the arc of the circle. Thirdly, the final value of the downwash as $t \to \infty$ is the same as the value before turning, for $\alpha < \pi$. When $\alpha = \pi$ the wing doubles back on itself and remains a non-dimensional distance 2 away from its original path, and so the final value for V_{3D} is different from the initial value. For $\alpha > \pi$ the wing will eventually cross its own wake rendering our solution invalid. The perturbation to the downwash due to turning (which reaches 30% for $\alpha = \pi$ and more for $\alpha > \pi$) is due entirely to the wake not lying in the plane of flight. A quasi-steady theory, which assumes the wake to always lie in the flight plane and does not account for the double-integral terms in (5.1), would predict a constant value for the downwash.

In the following section we shall briefly and qualitatively discuss the effects of wing thickness.

6. The effects of wing thickness – a dipole distribution

The preceding is valid for a thick wing in all but two respects; the thin-wing assumption has been used in (i) ignoring the dipole effect in ϕ_2 and avoiding the complexities arising from W^*, and (ii), replacing the vortex wake by the path of the trailing edge.

When the thickness of the wing is of the same order as the chord, the dipole of strength $a_{-1}\,\mathrm{Re}\{a_1\,\mathrm{e}^{2i\gamma}\dot{\bar{\chi}} - a_{-1}\dot{\chi}\}$ appears in the outer potential at $O(A^{-2})$ i.e. in ϕ_2. Thus ϕ_2 is to match with a line distribution of dipoles, and this is the same problem as in the classical theory of lateral flow past a slender body (Thwaites 1960). If we denote the complex dipole strength $a_{-1}\{a_1\,\mathrm{e}^{2i\gamma}\dot{\bar{\chi}} - a_{-1}\dot{\chi}\}$ by $a+ib$, then ϕ_2 will contain a new term given by

$$\frac{1}{2}\int_{-1}^{1}\frac{a(s,t)\,x+b(s,t)\,y}{\{x^2+y^2+(z-s)^2\}^{\frac{3}{2}}}\,\mathrm{d}s.$$

Note that this does not depend upon the history of the motion. As before we may expand this term for small x and y, with the result that the inner potential W must match with a term

$$A^{-3}\left\{\bar{I}_3^*\,w+\frac{I_3^*}{w}\right\}-\tfrac{1}{4}A^{-3}\left\{x\frac{\partial^2}{\partial z^2}a+y\frac{\partial^2}{\partial z^2}b\right\}\ln(x^2+y^2), \tag{6.1}$$

where

$$\frac{I_3^*}{a_{-1}}=\frac{1}{2}\int_{-1}^{1}\left\{a^*(s,t)-a^*(z,t)-(s-z)\frac{\partial a^*}{\partial z}(z,t)-\tfrac{1}{2}(s-z)^2\frac{\partial^2 a^*}{\partial z^2}(z,t)\right\}\Big/(s-z)^3\,\mathrm{d}s$$
$$+\frac{1}{4}\frac{\partial^2}{\partial z^2}\{a^*(z,t)\,[\ln(4A^2(1-z^2))-2]\} \tag{6.2}$$

and $a^* = a+ib$.

With the following definitions for $G(s,z)$ and $H(s,z)$:

$$w = \frac{\partial G}{\partial s},$$

and

$$\frac{1}{w} = \frac{\partial H}{\partial s},$$

we may solve (3.3) to give

$$W^* = -\tfrac{1}{4}I_1\,\bar{s}\frac{\partial^2}{\partial z^2}\{a_{-1}(G+H)-a_0\,s\}+p(s)$$

(assuming that γ is independent of z, that is there is no twist in the wing), where p is an analytic function to be determined from the tangency condition and the behaviour at infinity given by (6.1).

W^* will contain downwash terms and will therefore require a further circulation term in order to satisfy the Kutta condition. For a wing of elliptic planform with a^* proportional to $(1-z^2)$ the integral term in (6.2) is identically zero, and the remaining term is unbounded at the wing tips. However, unlike the non-uniformity encountered previously, this non-uniformity leads to logarithmically unbounded total loading on the wing, which must be removed by constructing a solution for the potential valid near the tips.

Also, when the wing is thick, a better approximation is needed for the wake position, although only near the wing (for distances of the order of the chord). To solve the inner problem exactly the wake must be perturbed by the local velocities of all orders. However, this is a prohibitively difficult analysis and to a first approximation it would be sufficient to consider only the local velocities induced by the uniform flow past the thick aerofoil (previously, for the thin wing, such velocities were a small perturbation to a uniform flow and so the wake was taken to be the path of the trailing edge – this cannot be justified for the thick wing). Since the streaklines of flow past a circle are simple to calculate, even in unsteady motion, such an analysis as outlined above is possible when the conformal mapping of the cross-section is known. An investigation along these lines is necessary and this problem will not be discussed further in this paper.

7. Forces on the wing

Returning to the case of the thin wing and applying the Kutta condition to $O(A^{-3})$ yields

$$\Gamma_{3_a} = 2a_{-1}\Gamma_{2_t}\,\mathrm{Re}\left\{\frac{\mathrm{e}^{\mathrm{i}\gamma(t)}}{\dot{\chi}(t)}\right\} \tag{7.1}$$

and
$$\Gamma_3 = 2\,\mathrm{Im}\{I_3\}-\frac{a_{-1}}{\pi}\mathrm{Re}\left\{\mathrm{e}^{\mathrm{i}\gamma(t)}\int_{t_0^-}^{t}\left[\frac{\Gamma_{2_\tau}(\tau,z)}{\chi(t)-\chi(\tau)}-\frac{\Gamma_{2_t}(t,z)}{(t-\tau)\,\dot{\chi}(t)}\right]\mathrm{d}\tau\right\}$$
$$+\frac{\Gamma_{2_t}}{2\pi}\int_0^\infty\left[\frac{2+h(\tau,t)+\bar{h}(\tau,t)}{(1+h(\tau,t)(1+\bar{h}(\tau,t))}-\frac{2H(\tau-t+t_0)\,a_{-1}}{\tau}\mathrm{Re}\left\{\frac{\mathrm{e}^{\mathrm{i}\gamma(t)}}{\dot{\chi}(t)}\right\}\right]\mathrm{d}\tau. \tag{7.2}$$

This completes the solution of the thin-wing inner problem to $O(A^{-3})$.

The forces on the wing may be written as the integral, over the surface, of the pressure in the direction of the normal, that is

$$F_x - \mathrm{i}F_y = \mathrm{i}A^{-1}\rho\Omega^2 l^4 \int_{|w|=1}\int_{-1}^{1} p\frac{\mathrm{d}s}{\mathrm{d}w}\mathrm{d}\eta\,\mathrm{d}w,$$

where $$p = \phi'_t + A\dot{\xi}\phi'_{x'} + A\dot{\zeta}\phi'_{y'} - \tfrac{1}{2}A^2(\phi'^2_{x'} + \phi'^2_{y'} + A^{-2}\phi'^2_{\bar{z}})$$

and $\phi' = \mathrm{Re}\{W\}$, F_x and F_y being in the x- and y-directions respectively.

If $s = a_{-1}(w+1/w) + O(A^{-1})$ then ϕ'_z and W^* can have at most an $O(A^{-4})$ effect on the forces. We can therefore take W to be a two-dimensional potential with z as a parameter and apply Milne-Thomson's extension of Blasius' theorem (Milne-Thomson 1938) to give the forces on the wing correct to $O(A^{-3})$. In inner non-dimensional variables the expression for the forces becomes

$$F_x - \mathrm{i}F_y = \tfrac{1}{2}\mathrm{i}\rho\Omega^2 l^4 A \int_{-1}^{1}\int_{|w|=1} \mathrm{e}^{-\mathrm{i}\gamma}\left(\frac{\mathrm{d}W}{\mathrm{d}w}\right)^2 \frac{\mathrm{d}w}{\mathrm{d}s}\mathrm{d}w\,\mathrm{d}\eta + \mathrm{i}\rho\Omega^2 l^4 A^{-1}$$

$$\times \int_{-1}^{1} \mathrm{e}^{-\mathrm{i}\gamma}\frac{\partial}{\partial t}\overline{\int_{|w|=1} W\frac{\mathrm{d}s}{\mathrm{d}w}\mathrm{d}w}\,\mathrm{d}\eta - \rho\Omega^2 l^4 A^{-1}\int_{-1}^{1}\dot{\gamma}\overline{\int_{|w|=1} s\frac{\mathrm{d}W}{\mathrm{d}w}\mathrm{d}w}\,\mathrm{d}\eta$$

$$+ \rho\Omega^2 l(\ddot{\xi} - \mathrm{i}\ddot{\zeta})V - 2\pi\mathrm{i}\Omega l^2(\dot{\xi} - \mathrm{i}\dot{\zeta})\int_{-1}^{1} K\,\mathrm{d}\eta,$$

where V is the volume of the wing and K is the local circulation. Upon substituting for W and evaluating contributions from poles the expression for the forces becomes

$$F_x - \mathrm{i}F_y = \rho\Omega^2 l(\ddot{\xi} - \mathrm{i}\ddot{\zeta})V + 2\pi\rho\Omega^2 l^4 \int_{-1}^{1} a_{-1}\frac{\partial}{\partial t}\{\mathrm{e}^{-\mathrm{i}\gamma}(a_1\bar{I}_1 - a_{-1}\bar{I}_1)\}\,\mathrm{d}\eta$$

$$-\mathrm{i}\rho\Omega^2 l^4 A^{-3}\int_{-1}^{1} \Gamma_{2_t}\mathrm{e}^{-\mathrm{i}\gamma}\overline{f(-1)}\,\mathrm{d}\eta$$

$$-\mathrm{i}\rho\Omega^2 l^4(\dot{\xi} - \mathrm{i}\dot{\zeta})\int_{-1}^{1}\{A^{-2}\Gamma_2 + A^{-3}\ln A\,\Gamma_{3_a} + A^{-3}\Gamma_3\}\,\mathrm{d}\eta + o(A^{-3}),$$

where Γ_2, Γ_{3_a} and Γ_3 are given by (5.2), (7.1) and (7.2) respectively.

We note that, as in the steady case, three-dimensional effects will first appear in the drag term at $O(\Gamma^2)$ i.e. $O(A^{-4})$. This term has not been calculated in this paper, and would necessitate finding the $O(A^{-4})$-term in the inner expansion. As has already been mentioned, this term will include effects due to the self-interaction of the wake, and is beyond the scope of this paper. Even if wake distortion is ignored, the problem of calculating the $O(A^{-4})$-inner term becomes prohibitively complicated when the motion is unsteady. Unfortunately, this inability to easily find the leading-order drag limits the practical application of the results (for example the two cases examined in §5) since the power needed for acceleration cannot be found without difficulty. Nevertheless, a qualitative interpretation can be placed on the results and as a rough estimation it may be possible to say that an increase (decrease) in downwash would produce an increase (decrease) in drag as in the steady case. However, this possibility needs further investigation. For example, when the wing moves through the arc of a circle (§5) the downwash is seen to decrease. This does not necessarily mean that the wing needs less power to turn.

8. Concluding remarks

In this paper we have considered the problem of a wing of high aspect ratio moving with three degrees of freedom. The method of matched asymptotic expansions has been used to find the inner and outer velocity potentials. Matching of these two potentials yields an expression for the downwash due to spanwise variation in local circulation. Two computed examples have been given. First, the spanwise variation in downwash is shown for an impulsively started wing. Secondly, the variation of mid-span downwash with time is given when the wing moves along a path that is curved in space.

This work extends that of Ahmadi & Widnall (1985) to include an extra degree of freedom in the path of the wing and allows an arbitrary time-variation.

We have given a qualitative description of the effects of wing thickness on the flow, downwash and lift. The effect on the latter is to include an instantaneous, as opposed to history-dependent, term.

Finally, expressions for the forces on the wing are given but it is noted that the drag force due to spanwise variations is affected by the self-interaction of the wake. It may nevertheless be possible to qualitatively describe the drag.

Further work is needed to include swept planform, flexibility of chord and span, ground effect (for an aircraft during take-off and landing) and further degrees of freedom in the wing motion.

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Appendix

We shall consider the limit of (4.4) with $x = A^{-1}x', y = A^{-1}y'$ as $A \to \infty$. First, integrate (4.4) by parts to give

$$\int_{-1}^{1}\int_{t_0^-}^{t} \frac{(\eta-z)\,\Gamma_{2_\eta}(\tau,\eta)\,[\dot{\xi}(\tau)\,(A^{-1}y'-\zeta(\tau)+\zeta(t))-\dot{\zeta}(\tau)\,(A^{-1}x'-\xi(\tau)+\xi(t))]}{\Delta_1\,\Delta_2^{\frac{1}{2}}}\,\mathrm{d}\tau\,\mathrm{d}\eta,$$

where $$\Delta_1 = (A^{-1}x'-\xi(\tau)+\xi(t))^2+(A^{-1}y'-\zeta(\tau)+\zeta(t))^2$$

and $$\Delta_2 = \Delta_1+(\eta-z)^2.$$

To this we may add and subtract a term

$$A^{-1}(\dot{\xi}(t)\,y'-\dot{\zeta}(t)\,x')\int_{-1}^{1}\int_{t_0^-}^{t}\frac{(\eta-z)\{\Gamma_{2_\eta}(t,\eta)+(\tau-t)\,\Gamma_{2_{\eta t}}(t,\eta)\}\,\mathrm{d}\tau\,\mathrm{d}\eta}{\bar{\Delta}_1\,\bar{\Delta}_2^{\frac{1}{2}}},$$

where $$\bar{\Delta}_1 = (A^{-1}x'+\dot{\xi}(t)\,(t-\tau))^2+(A^{-1}y'+\dot{\zeta}(t)\,(t-\tau))^2$$

and $$\bar{\Delta}_2 = \bar{\Delta}_1+(\eta-z)^2,$$

to give a non-singular integral which may be expanded for large A plus an integral which may be evaluated exactly. Hence we have

$$
\begin{aligned}
A^{-1}\int_{-1}^{1}\int_{t_0^-}^{t}\Bigg[&\frac{(\eta-z)\,\Gamma_{2_\eta}(\tau,\eta)}{\Delta_{1,0}\Delta_{2,0}^{\frac{1}{2}}}\Big\{y'\dot{\xi}(\tau)-x'\dot{\zeta}(\tau)\\
&-\{x'(\xi(t)-\xi(\tau))+y'(\zeta(t)-\zeta(\tau))\}\\
&\times\{\dot{\xi}(\tau)(\zeta(t)-\zeta(\tau))-\dot{\zeta}(\tau)(\xi(t)-\xi(\tau))\}\left\{\frac{2}{\Delta_{1,0}}+\frac{1}{\Delta_{2,0}}\right\}\Big\}\\
&-\frac{(\eta-z)(y'\dot{\xi}(t)-x'\dot{\zeta}(t))\{\Gamma_{2_\eta}(t,\eta)+(\tau-t)\,\Gamma_{2_{\eta t}}(t,\eta)\}}{\bar{\Delta}_{1,0}\bar{\Delta}_{2,0}^{\frac{1}{2}}}\Bigg]\mathrm{d}\tau\,\mathrm{d}\eta\\
&+A^{-1}(\dot{\xi}(t)\,y'-\dot{\zeta}(t)\,x')\int_{-1}^{1}\int_{t_0^-}^{t}\frac{(\eta-z)\{\Gamma_{2_\eta}(t,\eta)+(\tau-t)\,\Gamma_{2_{\eta t}}(t,\eta)\}}{\bar{\Delta}_1\bar{\Delta}_2^{\frac{1}{2}}}\mathrm{d}\tau\,\mathrm{d}\eta,
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta_{1,0}&=(\xi(t)-\xi(\tau))^2+(\zeta(t)-\zeta(\tau))^2,\\
\Delta_{2,0}&=\Delta_{1,0}+(\eta-z)^2,\\
\bar{\Delta}_{1,0}&=(\dot{\xi}(t)^2+\dot{\zeta}(t)^2)(t-\tau)^2,
\end{aligned}
$$

and

$$\bar{\Delta}_{2,0}=\bar{\Delta}_{1,0}+(\eta-z)^2.$$

The final term may be integrated with respect to u exactly (Gradshteyn & Ryzhik 1980, p. 89).

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